Products of subgroups

Def: Let H and K be subgroups of G. Define

$$HK = \{hk \mid h \in H, k \in K\}.$$

Note that $H \subseteq HK$ and $K \subseteq HK$, but HK is not necessarily a subgroup of G — eg. $\langle(1\,2)\rangle\langle(2\,3)\rangle = \{1, (1\,2), (2\,3), (1\,2\,3)\}$, which has order 4 and is thus not a subgroup of S_3.

If G is abelian, then $(hk)(h'k')^{-1} = (hh'^{-1})(kk'^{-1}) \in HK$, so $HK \leq G$. In fact, it is a subgroup in a more general setting:

Thm: $HK \leq G \iff HK = KH$.

Pf: If $HK \leq G$, then $\forall\ h \in H, k \in K$, we know $h \cdot 1 = h \in HK$ and $1 \cdot k = k \in HK$, so $kh \in HK$. Thus $KH \subseteq HK$.

For the reverse containment, if $hk \in HK$, then $(hk)^{-1} \in HK$, so $(hk)^{-1} = h_1 k_1$, some $h_1 \in H$, $k_1 \in K$

Thus $hk = (h_1 k_1)^{-1} = k_1^{-1} h_1^{-1} \in KH$. So $HK = KH$.

For the converse, assume $HK = KH$.

Let $a, b \in HK$. We want to show $ab^{-1} \in HK$

Let $a = h_1k_1$, $b = h_2k_2$. Then $ab^{-1} = h_1 \underbrace{k_1 k_2^{-1} h_2^{-1}}_{\in\, KH = HK} = h_1 h_3 k_3 \in HK$.

$\Rightarrow HK \leq G$. $\square$

(Note that $HK = KH$ does <u>not</u> mean elements of H commute w/ elements of K. It just means we can write hk as $hk = k'h'$, for some $k' \in K$, $h' \in H$.)

<u>Cor</u>: If $K \trianglelefteq G$, then $HK \leq G$ for any $H \leq G$.

<u>Pf</u>: Assume $K \trianglelefteq G$. $HK = \bigcup_{h \in H} hK \overset{\text{normality!}}{=} \bigcup_{h \in H} Kh = KH$. $\Rightarrow HK \leq G$. $\square$

If G is finite, how many elements does HK have?

HK is the union of cosets, but some of those cosets may be equal.

$$\begin{aligned} h_1K = h_2K &\Leftrightarrow h_1 = h_2k, \text{ some } k \in K \\ &\Leftrightarrow h_2^{-1}h_1 \in K. \\ &\Leftrightarrow h_2^{-1}h_1 \in H \cap K \Leftrightarrow h_1(H\cap K) = h_2(H \cap K). \end{aligned}$$

So the number of <u>distinct</u> cosets in the union is

$\dfrac{|H|}{|H \cap K|}$ by Lagrange's theorem. Since each coset has $|K|$ elements,

$$\boxed{|HK| = \frac{|H||K|}{|H \cap K|}}.$$